NEW YORK UNIVERSITY
COURANT INSTITUTE - LIBRARY
251 Mercer St. New York, N.Y. 10012

AFOSR TN-56-270
ASTIA DOCUMENT No. AD 88990

NEW YORK UNIVERSITY
Institute of Mathematical Sciences
Division of Electromagnetic Research

RESEARCH REPORT No. BR-19

On the Functions of the Parabolic Cylinder

DAVID I. EPSTEIN

Mathematics Division
Air Force Office of Scientific Research
Contract No. AF 18(600)-367
File No. 2.2

JUNE, 1956

c. 2

AFOSR TN-56-270
ASTIA DOCUMENT No. AD 88990

NEW YORK UNIVERSITY
Institute of Mathematical Sciences
Division of Electromagnetic Research

Research Report No. BR-19

ON THE FUNCTIONS OF THE PARABOLIC CYLINDER

David I. Epstein

David I. Epstein
David I. Epstein

Morris Kline
Morris Kline
Project Director

June, 1956

The research reported in this document has been made possible through support and sponsorship extended by H.Q., Air Force Office of Scientific Research, Air Research and Development Command, Baltimore, Maryland, under Contract No. AF 18(600)-367, File No. 2.2. It is published for technical information only and does not necessarily represent recommendations or conclusions of the sponsoring agency.

New York, 1956

NEW YORK UNIVERSITY
COURANT INSTITUTE - LIBRARY
251 Mercer St. New York, N.Y. 10012

c.2

Abstract

An addition theorem is obtained for the separated solutions of the reduced wave equation in parabolic cylinder coordinates; it is based on the property that the reduced wave equation is invariant under translations and rotations of the coordinate system. An interesting feature of the theorem is that the basic set of the separated solutions that are regular and one-valued everywhere is split into two disjoint subsets such that the addition theorem for the functions of one set involves functions of that set only.

A brief discussion is devoted to the problem of expanding any separated one-valued regular solution in terms of the solutions belonging to the basic set.

Certain integrals which arise in diffraction problems and involve functions of the parabolic cylinder are evaluated asymptotically for large values of a parameter. The parameter in question is the reciprocal wavelength of the incident electromagnetic field.

Table of Contents

Page

Introduction 1

Part I. The Addition Theorems

1. Preliminaries 2
2. Notations and auxiliary formulas 4
3. Special one-valued solutions 8
4. Addition theorem for translation 10
5. Addition theorem for rotation 12
6. Expansion problem for ω_ν 14

Part II. A Diffraction Problem

1. Formulation of the problem 16
2. Remarks on the case of short wavelength 21

Bibliography 24

Distribution List 25

Introduction

The reduced wave equation $\Delta u + k^2 u = 0$ is invariant under translations and rotations of the coordinate system. Within the set of all solutions, the family of solutions that are everywhere regular and one-valued must form an invariant subset under translations and rotations. Now suppose that in a certain coordinate system separation of the variables is possible. Then a translation or rotation applied to the variables of a separated one-valued and regular solution leads to a solution which, though not separated, is regular and one-valued again. This last solution must therefore be expressible as a linear combination of the separated, regular and one-valued solutions provided that these are complete in the space of the regular and one-valued solutions. This proves the existence of addition theorems for functions arising from the separation of variables for $\Delta u + k^2 u = 0$. In the classical case of polar coordinates, the resulting regular and one-valued functions are, in two dimensions, the functions

$$J_n(k\rho)\, e^{in\emptyset}, \qquad n = 0, \pm 1, \pm 2, \ldots .$$

The fact that the non-regular solutions $J_\nu(k\rho)e^{i\nu\emptyset}$ and $H_\nu^{(1)}(k\rho)e^{i\nu\emptyset}$ also have an addition theorem can be derived from theorems about the behavior of these functions in the infinite part of the plane. The analogous addition theorems for Mathieu functions and spheroidal wave functions have been derived by Meixner and Schäfke [6].

In the case of the coordinate system of the parabolic cylinder, the separated solutions that are everywhere regular and one-valued are found easily, but their completeness shows a rather curious deficiency. With two exceptions, all plane waves can be expanded in convergent series of these regular solutions. The set of all regular, separated solutions splits into two subsets such that the expansion of a plane wave involves the functions of one set only. Consequently, the addition theorems for the regular solutions are such that they involve functions either of one set or of the other one.

The functions of the parabolic cylinder have previously been applied to the diffraction problem arising from the presence of an electromagnetic line source in the interior of a parabolic cylinder of perfect conductivity (see Magnus [1] and Buchholz [2]). The procedure used in these papers has been criticized by Greenberg, et al. [3]. It is shown in the present paper that this criticism is not relevant. Also, some formulas are derived which describe the asymptotic behavior of the diffracted field in a part of the interior of the cylinder for a wavelength which is small compared with the focal length of the parabola.

Part I. The Addition Theorems

1. Preliminaries

We shall be concerned in the first part of this paper with the problem of addition theorems for certain functions which are combinations of the parabolic cylinder functions.

Generally speaking, we consider a coordinate system, ρ, σ for which the equation

$$(1.1) \qquad \triangle u + k^2 u = 0$$

admits separation of variables. Then, certain solutions of (1.1) will be of the form

$$(1.2) \qquad u = \omega_\nu(\rho,\sigma) = R_\nu(\rho)S_\nu(\sigma) ,$$

where ν is a complex parameter. In general, the $\omega_\nu(\rho,\sigma)$ are regular and one-valued only for a subset of the set of complex numbers; for the parabolic cylinder functions this subset coincides with the set of all integers, n.

Suppose now that the x,y-plane is subjected to the general transformation

$$(1.3) \qquad \begin{cases} x' = x\cos\alpha + y\sin\alpha + x_o , \\ y' = -x\sin\alpha + y\cos\alpha + y_o . \end{cases}$$

Then the coordinates $\rho = \rho(x,y)$ and $\sigma = \sigma(x,y)$ are transformed into $\rho' = \rho(x',y')$ and $\sigma' = \sigma(x',y')$.

Write

$$\omega_\nu' = \omega_\nu(\rho', \sigma') = R_\nu(\rho')\, S_\nu(\sigma').$$

One of our principal problems will be to express these ω_ν' in terms of the $\omega_\nu(\rho,\sigma)$.

Another question that we decide is that of the completeness of the set of functions ω_n that are everywhere one-valued and regular. This is done by observing that all but two plane waves can be expanded in terms of the ω_n. It is well known that the plane waves form a complete system in the sense that a regular, one-valued solution of (1.1) can be approximated in any finite part of the plane with an arbitrarily small error term by a linear combination of plane waves.

If we specialize (1.3) to the transformation

$$(1.3)' \qquad \begin{cases} x' = x + x_o \\ y' = y + y_o \end{cases}$$

we obtain a formula of the type

$$(1.4) \qquad \omega_n(\rho',\sigma') = \sum_{\ell,m} \gamma_{\ell,m}^{(n)}\, \omega_\ell(\rho,\sigma)\, \omega_m(\rho_o,\sigma_o),$$

where $\rho_o = \rho(x_o,y_o)$, $\sigma_o = \sigma(x_o,y_o)$. The range of the ℓ, m,n is the group of integers, and the $\gamma_{\ell,m}^{(n)}$ are numerical factors.

For the rotation

$$(1.3)'' \qquad \begin{cases} x' = x\cos\alpha + y\sin\alpha \\ y' = -x\sin\alpha + y\cos\alpha \end{cases}$$

we obtain a relation of the form

$$(1.5) \qquad \omega_n(\rho',\sigma') = \sum A_m^{(n)}(\alpha)\, \omega_m(\rho,\sigma) ,$$

the $A_m^{(n)}$ depending only on α.

2. Notations and auxiliary formulas

The equation which concerns us is

$$(2.1) \qquad \triangle u + k^2 u = 0, \qquad u = u(x,y).$$

If we introduce the coordinates of the parabolic cylinder

$$(2.2) \qquad (\xi + i\eta)^2 = 2(x + iy),$$

or equivalently,

$$(2.3) \qquad \begin{cases} x = \frac{1}{2}(\xi^2 - \eta^2), \\ y = \xi\eta , \end{cases}$$

equation (2.1) becomes

$$(2.4) \qquad \frac{\partial^2 u}{\partial\xi^2} + \frac{\partial^2 u}{\partial\eta^2} + k^2(\xi^2 + \eta^2)u = 0.$$

Putting

$$(2.5) \qquad u = f_1(\xi)\, f_2(\eta),$$

we obtain for f_1 and f_2 the following equations:

$$(2.6a) \qquad \frac{d^2 f_1}{d\xi^2} + (w + k^2\xi^2)\, f_1 = 0,$$

$$(2.6b) \qquad \frac{d^2 f_2}{d\eta^2} + (-w + k^2\eta^2)\, f_2 = 0.$$

Here w denotes an arbitrary (complex) constant.

Particular solutions of (2.6a) and (2.6b) are

$$f_1(\xi) = D_\nu\left(\sqrt{k}\,(1+i)\xi\right),\ D_\nu\left(-\sqrt{k}\,(1+i)\xi\right),\ D_{-\nu-1}\left(i\sqrt{k}\,(1+i)\xi\right),$$

(2.7a)

$$D_{-\nu-1}\left(-i\sqrt{k}\,(1+i)\xi\right),$$

$$f_2(\eta) = D_{-\nu-1}\left(\sqrt{k}\,(1+i)\eta\right),\ D_{-\nu-1}\left(-\sqrt{k}\,(1+i)\eta\right),\ D_\nu\left(i\sqrt{k}\,(1+i)\eta\right),$$

(2.7b)

$$D_\nu\left(-i\sqrt{k}\,(1+i)\eta\right),$$

where $\nu = -\frac{1}{2} - \frac{iw}{2k}$.

The functions $D_\nu(z)$ can be defined by a power series, as in [1], vol. II, p. 117:

$$D_\nu(z) = 2^{(\nu-1)/2} e^{-z^2/4} \cdot z \cdot \Psi\left(\frac{1}{2} - \frac{\nu}{2}; \frac{3}{2}; \frac{z^2}{2}\right), \tag{2.8}$$

or

$$D_\nu(z) = 2^{\nu/2} e^{-z^2/4}\left[\frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2} - \frac{\nu}{2})}\,\Phi\left(-\frac{\nu}{2}; \frac{1}{2}; \frac{z^2}{2}\right) + \frac{z}{\sqrt{2}}\,\frac{\Gamma(-\frac{1}{2})}{\Gamma(-\frac{\nu}{2})}\right.$$
$$\left.\cdot\ \Phi\left(\frac{1}{2} - \frac{\nu}{2}; \frac{3}{2}; \frac{z^2}{2}\right)\right]. \tag{2.9}$$

The functions Ψ and Φ are explained in [1], vol. I, chapter 6. The function Φ is sometimes denoted by

$${}_1F_1(a;c;z) = 1 + \frac{a}{c}\cdot z + \frac{a(a+1)}{c(c+1)}\,\frac{z^2}{2!} + \dots .$$

It is a consequence of (2.9) that for $\nu = n = 0, 1, 2, \dots$,

$$D_n(z) = e^{-z^2/4}\, 2^{-n/2} \cdot H_n(2^{-1/2}z),$$

where H_n is the Hermite polynomial of degree n.

Between the functions D_ν and $D_{-\nu-1}$ there exists the following linear relation:

$$(2.10) \qquad D_\nu(z) = \frac{\Gamma(\nu+1)}{\sqrt{2\pi}} \left[e^{\nu\pi i/2} D_{-\nu-1}(iz) + e^{-\nu\pi i/2} D_{-\nu-1}(-iz) \right] .$$

An important relation which is a consequence of (2.10) is obtained by putting $\nu = 2n$ and replacing z by iz:

$$(2.11a) \qquad \frac{(-1)^n \sqrt{2\pi}}{(2n)!} D_{2n}(iz) = D_{-2n-1}(-z) + D_{-2n-1}(z).$$

By putting $\nu = 2n+1$ and $z = -i\gamma\eta$ we obtain

$$(2.11b) \qquad D_{2n+1}(-i\gamma\eta) = \frac{(-1)^n i(2n+1)!}{\sqrt{2\pi}} \left[D_{-2n-2}(\gamma\eta) - D_{-2n-2}(-\gamma\eta) \right] .$$

These two relations enable us to write the functions

$$(2.12) \qquad \omega_\nu(\xi, \eta) = D_\nu(\gamma\xi) D_{-\nu-1}(\gamma\eta) + D_\nu(-\gamma\xi) D_{-\nu-1}(-\gamma\eta),$$

which appear in subsequent work, in a simpler form when ν is an integer. Thus,

$$(2.13) \qquad \begin{aligned} \omega_{2n}(\xi, \eta) &= D_{2n}(\gamma\xi) D_{-2n-1}(\gamma\eta) + D_{2n}(-\gamma\xi) D_{-2n-1}(-\gamma\eta) \\ &= D_{2n}(\gamma\xi) \left[D_{-2n-1}(\gamma\eta) + D_{-2n-1}(-\gamma\eta) \right] \\ & \frac{\sqrt{2\pi}\,(-1)^n}{(2n)!} D_{2n}(\gamma\xi) D_{2n}(i\gamma\eta), \quad . \end{aligned}$$

since $D_{2n}(z) = D_{2n}(-z)$. Similarly, by the use of the (2.11b) we find

$$(2.14) \qquad \omega_{2n+1}(\xi, \eta) = \frac{i\sqrt{2\pi}\,(-1)^{n+1}}{(2n+1)!} D_{2n+1}(\gamma\xi) D_{2n+1}(-i\gamma\eta).$$

In the same way we can simplify the expressions ω_{-2n} :

$$\omega_{-2n} = D_{-2n}(\gamma\xi)D_{+2n-1}(\gamma\eta) + D_{-2n}(-\gamma\xi)D_{+2n-1}(-\gamma\eta),$$

and

$$\omega_{-2n-1} = D_{-2n-1}(\gamma\xi)D_{2n}(\gamma\eta) + D_{-2n-1}(-\gamma\xi)D_{2n}(-\gamma\eta).$$

It will be necessary, in connection with the expressions ω_{2n}, ω_{2n+1}, ω_{-2n}, ω_{-2n-1}, to consider the asymptotic expansions for the products $D_n(\gamma\xi)D_{-n-1}(\gamma\eta)$, $D_{2n}(\gamma\xi)D_{2n}(i\gamma\eta)$, and $D_{2n+1}(\gamma\xi)D_{2n+1}(-i\gamma\eta)$, for large n. For this purpose we use

$$D_\nu(z) \approx 2^{-1/2} \exp\left\{ \frac{\nu}{2}\log(-\nu) - \frac{\nu}{2} - (-\nu)^{1/2} z \right\}, \tag{2.15}$$

where $|z|$ is bounded and $|\arg(-\nu)| \leq \frac{\pi}{2}$, and $|\nu| \to \infty$; we use also (2.11a) and (2.11b). If it is not true that $|\arg(-\nu)| \leq \frac{\pi}{2}$, then we must first use (2.10). From these relations we have for $|\nu| \to \infty$:

$$\begin{aligned} D_n(\gamma\xi)D_{-n-1}(\gamma\eta) \approx \frac{2^{-1}}{\sqrt{n+1}} \Big[&\exp\left\{ -\gamma(n+1)^{1/2}(\eta + i\xi) + \frac{n\pi i}{2} \right\} \\ &+ \exp\left\{ \gamma(n+1)^{1/2}(i\xi - \eta) - \frac{n\pi i}{2} \right\} \Big] \end{aligned} \tag{2.16}$$

$$\begin{aligned} D_{2n}(\gamma\xi)D_{2n}(i\gamma\eta) \approx \frac{(2n)!}{2\sqrt{2\pi}\sqrt{2n+1}} \Big[&\exp\left\{ (2n+1)^{1/2}\gamma(\eta - i\xi) \right\} \\ &+ \exp\left\{ -(2n+1)^{1/2}\gamma(i\xi + \eta) \right\} \\ &+ \exp\left\{ (2n+1)^{1/2}\gamma(i\xi + \eta) \right\} \\ &+ \exp\left\{ (2n+1)^{1/2}\gamma(i\xi - \eta) \right\} \Big] \end{aligned} \tag{2.17}$$

and

$$D_{2n+1}(\gamma\xi)D_{2n+1}(-i\gamma\eta) \approx \frac{(2n+1)!}{2\sqrt{2\pi}\sqrt{2n+2}}\Big[\exp\Big\{-(2n+2)^{1/2}\gamma(\eta+i\xi)\Big\}+\exp\Big\{-(2n+2)^{1/2}\gamma(i\xi-\eta)\Big\}$$

$$+\exp\Big\{(2n+2)^{1/2}\gamma(i\xi-\eta)\Big\}+\exp\Big\{(2n+2)^{1/2}\gamma(i\xi+\eta)\Big\}\Big]. \tag{2.18}$$

We must also consider the asymptotic expansion of the product

$$D_{\nu}(\gamma\xi)\,D_{-\nu-1}(\gamma\eta) \tag{2.19}$$

for $\nu = -\frac{1}{2} + i\lambda$, λ real and tending to $\pm\infty$. For $\lambda \to \infty$,

$$D_{-1/2+i\lambda}(\gamma\xi)D_{-1/2-i\lambda}(\gamma\eta) \approx 2^{-1}\exp\Big\{-\frac{1}{2}\log\lambda + \frac{\pi\lambda}{2} - (\frac{\lambda}{2})^{1/2}\gamma(\xi+\eta)$$

$$+\, i\Big(-\lambda + (\frac{\lambda}{2})^{1/2}\gamma(\xi+\eta)\Big)\Big\}. \tag{2.20}$$

The corresponding expression for $\lambda \to -\infty$ can be obtained by putting $\lambda = -t$ and letting $t \to +\infty$, or by interchanging ξ and η and letting $\lambda \to \infty$ as above.

3. Special one-valued solutions

Before entering upon a discussion of the addition theorems it will be of some value to remark on the equation of one-valuedness.

From the transformation (2.3) we see that a point in the (x,y)-system has two pairs of coordinates in the (ξ,η)-system, namely (ξ,η) and $(-\xi,-\eta)$. Thus, to say that the function $F(\xi,\eta)$ is one-valued in the (x,y)-plane means that $F(\xi,\eta) = F(-\xi,-\eta)$. Therefore, a product $f_1(\gamma\xi)f_2(\gamma\eta)$ is a one-valued function in the (x,y)-plane if and only if the functions f_1 and f_2 are both even or both odd.

Given $f_1(\gamma\xi)$ and $f_2(\gamma\eta)$, then clearly

$$f_1(\gamma\xi)f_2(\gamma\eta) + f_1(-\gamma\xi)f_2(-\gamma\eta) \tag{3.1}$$

is a one-valued solution of the wave equation, although one in which the variables are not completely separated. For certain purposes, for example Cherry's inversion formula ([1], vol. II, p. 124), these are the proper functions to use. In particular, we shall need the functions

$$\omega_\nu(\xi,\eta) = D_\nu(\gamma\xi)D_{-\nu-1}(\gamma\eta) + D_\nu(\gamma\xi)D_{-\nu-1}(-\gamma\eta). \tag{3.2}$$

They are fully separable for integer values of ν, according to (2.11) and (2.12).

That the functions (3.2) form a basis for the set of one-valued regular solutions of the wave equation follows from the generating function relation for the parabolic cylinder functions

$$\frac{e^{F(x,y,t)}}{\sqrt{1+t^2}} = \frac{\sqrt{\pi/2}}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{t^{-\nu-1}}{\sin(-\nu\pi)}\,\omega_\nu(x,iy)\,d\nu, \tag{3.3}$$

where

$$F(x,y,t) = \frac{1-t^2}{4(1+t^2)}(x^2+y^2) + \frac{itxy}{1+t^2}.$$

Then, if in (3.3) we replace x by $\sigma\xi$ and iy by $\sigma\eta$, where $\sigma = \sqrt{k}\,(1+i)$, we obtain

$$\frac{\sqrt{\pi/2}}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{t^{-\nu-1}}{\sin(-\nu\pi)}\,\omega_\nu(\sigma\xi,\sigma\eta)\,d\nu = \frac{1}{\sqrt{1+t^2}}\exp\left\{ik\left[\frac{1-t^2}{1+t^2}\frac{(\xi^2-\eta^2)}{2} + \frac{2t}{1+t^2}\xi\eta\right]\right\} \tag{3.4}$$

$$\equiv G(\xi,\eta,t),$$

where $-1 < c < 0$.

If we introduce an angle $\emptyset$ by means of

$$t = \tan\frac{\emptyset}{2},$$

then the right-hand side, G, of (3.4) is a plane wave incident at an angle $\emptyset$ to the x-axis.

We may apply the Cauchy residue theorem to the integral in (3.4), which has simple poles at $\nu = 0, \pm 1, \pm 2, \ldots$. If we close the path of integration to the left, the poles inside the contour are $\nu = -1, -2, \ldots$, and we obtain

$$(3.5) \qquad G(\xi,\eta,t) = \sqrt{\pi/2} \sum_{n=1}^{\infty} (-1)^n \omega_n(\xi,\eta) t^{-n-1}, \qquad (|t| > 1).$$

If we close the path of integration to the right, then the poles inside the contour are $\nu = 0, 1, 2, \ldots$, so that

$$(3.6) \qquad G(\xi, \eta, t) = \sqrt{\pi/2} \sum_{n=0}^{\infty} (-1)^n \omega_{-n-1} t^n, \qquad (|t| < 1).$$

In (3.5) and (3.6) the inequalities in parentheses indicate the range for which the series converge.

We can use the formulas for the asymptotic behavior of the $\omega_\nu(\sigma\xi,\sigma\eta)$ for large ν to show that deformation of path is permitted. Also, we can prove the convergence of (3.5) and (3.6) by means of (2.15) and (2.16). Moreover, (3.5) and (3.6) show that the plane waves can be expanded, except for those which are defined by $t = \pm 1$. The latter are the waves that are incident at an angle of 45° to the x-axis. For $t = \pm 1$, the series (3.5) and (3.6) diverge.

4. Addition theorem for translation

From the considerations of the previous section we are led to the addition theorem for translations.

We have

$$\begin{cases} \xi'^2 - \eta'^2 = \xi^2 - \eta^2 + (\xi_0^2 - \eta_0^2) \\ \xi'\eta' = \xi\eta + \xi_0\eta_0 . \end{cases}$$

From the definition of $G(\xi, \eta, t)$ we see that the following relation holds with respect to the above translation:

$$G(\xi',\eta',t) = \sqrt{1+t^2}\cdot G(\xi,\eta,t)\cdot G(\xi_o,\eta_o,t). \tag{4.1}$$

Now let us put

$$\sqrt{1+t^2} = \begin{cases} \sum_0^\infty \lambda_n t^n, & |t|<1 \\ \sum_0^\infty \lambda_n t^{-n+1}, & |t|>1. \end{cases}$$

Taking coefficients on both sides of (4.1) gives us

$$\sqrt{\pi/2}\sum_0^\infty (-1)^n\,\omega_{-n-1}(\xi',\eta')t^n = \left(\sum_0^\infty \lambda_n t^n\right)\cdot\sqrt{\pi/2}\cdot\left(\sum_0^\infty (-1)^n\,\omega_{-n-1}(\xi,\eta)t^n\right)$$

$$\cdot\sqrt{\pi/2}\cdot\left(\sum_0^\infty (-1)^n\,\omega_{-n-1}(\xi_o,\eta_o)t^n\right). \tag{4.2}$$

This may be written as

$$\sqrt{\pi/2}\,(-1)^n\,\omega_{-n-1}(\xi',\eta') = \frac{\pi}{2}\sum_{\substack{\rho+\sigma+\tau\\=n}} \lambda_\rho\,\omega_{-\sigma-1}(\xi,\eta)\omega_{-\tau-1}(\xi_o,\eta_o)(-1)^{\sigma+\tau}. \tag{4.3}$$

where $\rho, \sigma, \tau, n = 0, 1, 2, \ldots$.

Hence, the addition theorem for the ω_{-n-1}, $n = 0, 1, \ldots$, reads:

$$\omega_{-n-1}(\xi',\eta') = \sqrt{\pi/2}\sum_{\substack{\rho+\sigma+\tau\\=n}} (-1)^\rho\,\lambda_\rho\,\omega_{-\sigma-1}(\xi,\eta)\omega_{-\tau-1}(\xi_o,\eta_o). \tag{4.4}$$

The coefficients λ_ρ that appear above are defined by

$$\lambda_\rho = 0 \quad \text{if } \rho \text{ is an odd integer;}$$

$$\lambda_{2r} = \binom{\frac{1}{2}}{r} = \frac{(-1)^r(-\frac{1}{2})_r}{r!} \quad .$$

Equation (4.4) corresponds to the case $|t| < 1$. When $|t| > 1$,

$$\sqrt{\pi/2}\ \sum_{1}^{\infty} (-1)^n\, \omega_n(\xi',\eta')t^{-n+1} = \left(\sum_{1}^{\infty} \lambda_n t^{-n+t}\right)\cdot \sqrt{\pi/2}\cdot\left((-1)^n\, \omega_n(\xi,\eta)t^{-n+1}\right)$$

$$(4.5)\qquad \cdot\ \sqrt{\pi/2}\left(\sum_{1}^{\infty} (-1)^n\, \omega_n(\xi_o,\eta_o)t^{-n+1}\right).$$

Formula (4.5) provides us with the companion addition theorem for the ω_n, $n = 0, 1, 2, \ldots$:

$$(4.6)\qquad \omega_n(\xi',\eta') = \sqrt{\pi/2} \sum_{\substack{\rho+\sigma+\tau\\=n}} \lambda_\rho(-1)^\rho\, \omega_\sigma(\xi,\eta)\omega_\tau(\xi_o,\eta_o),$$

where $o, \sigma, \tau = 0, 1, 2, \ldots$.

Formulas (4.4) and (4.6) exhibit an interesting feature. In (4.4) only the functions ω_{-n-1} are related; in (4.5) only the functions ω_n are related.

5. Addition theorem for rotation

In this section the problem is to obtain an addition theorem for the rotation

$$(5.1)\qquad \begin{cases} \xi^* = \xi\cos\frac{\alpha}{2} - \eta\sin\frac{\alpha}{2} \\[2ex] \eta^* = \eta\cos\frac{\alpha}{2} + \xi\sin\frac{\alpha}{2} . \end{cases}$$

From the bilinear continuous generating function for the D_ν (cf. [1], vol. II, p. 126) with $x = \sigma\xi$, $iy = \sigma\eta$, where $\sigma = \sqrt{k}\,(1+i)$, we get

$$G(\xi,\eta,t) = \frac{1}{\sqrt{1+t^2}}\exp\left\{ik\left[\cos\phi\,\frac{\xi^2-\eta^2}{2} + \sin\phi\,\xi\eta\right]\right\}$$

$$(5.2)\qquad = \frac{\sqrt{\pi/2}}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{t^{-\nu-1}}{\sin(-\nu\pi)}\,\omega_\nu(\sigma\xi,\sigma\eta)\,d\nu .$$

The trigonometric functions appearing in (5.2) are explained by the substitution

$$(5.3)\quad \begin{cases} t = \tan\frac{\phi}{2}, \quad \text{from which it follows that} \\ \frac{1-t^2}{1+t^2} = \cos\phi, \quad \frac{2t}{1+t^2} = \sin\phi . \end{cases}$$

Also

$$(5.4)\qquad G(\xi^*, \eta^*, t) = \frac{1}{\sqrt{1+t^2}} \exp\left\{ ik\left[\cos(\phi-\alpha)\,\frac{\xi^2-\eta^2}{2} + \sin(\phi-\alpha)\xi\eta\right]\right\} .$$

Therefore

$$(5.5)\qquad G(\xi^*, \eta^*, t) = \frac{\sqrt{1+t^{*2}}}{\sqrt{1+t^2}} \cdot G(\xi, \eta, t^*),$$

where

$$t^* = \tan(\phi-\alpha) = \frac{\tan\phi - \tan\alpha}{1+\tan\phi\,\tan\alpha} .$$

If we put $\tan\alpha = A$, then we can write t^* as

$$(5.6)\qquad t^* = (-A)\left(\frac{1-\frac{t}{A}}{1+At}\right) .$$

If we apply the residue theorem to (5.2) with ξ, η replaced by ξ^*, η^* respectively, we obtain

$$(5.7)\qquad \begin{aligned} G(\xi^*, \eta^*, t) &= \sqrt{\pi/2} \sum_1^\infty (-1)^n\, \omega_{-n}(\xi^*, \eta^*) t^{n-1} \\ &= \sqrt{\pi/2}\, \frac{\sqrt{1+t^{*2}}}{\sqrt{1+t^2}} \sum_1^\infty (-1)^n\, \omega_{-n}(\xi, \eta) t^{*n-1} . \end{aligned}$$

Putting $n-1 = m$, we can write

$$(5.8)\qquad \sum_0^\infty (-1)^{m+1}\, \omega_{-m-1}(\xi^*, \eta^*) t^m = \frac{\sqrt{1+t^{*2}}}{\sqrt{1+t^2}} \sum_0^\infty (-1)^{m+1}\, \omega_{-m-1}(\xi, \eta) t^{*m} .$$

Now we must expand $\sqrt{1+t^{*2}} \big/ \sqrt{1+t^2}$ and t^{*m} on the right-hand side and then compare the coefficients of similar powers of t on both sides. We shall then have the required addition theorem.

From (5.6) we have

$$(5.9)\qquad \left(\frac{1+t^{*2}}{1+t^{2}}\right)^{1/2} = \frac{(1+A^2)^{1/2}}{1+At} = (1+A^2)^{1/2}\sum_{\ell=0}^{\infty}(-At)^{\ell} \quad ,$$

and, for $m > 0$, after some calculations:

$$(5.10)\qquad t^{*m} = (-A)^m \sum_{n=0}^{\infty} \frac{(m+n-1)!}{(m-1)!\,n!}\ {}_2F_1(-n,-m;1-n-m;-A^{-2})(-At)^n \ .$$

Here ${}_2F_1$ denotes the hypergeometric series. By substituting (5.9) and (5.10) in (5.8) and comparing coefficients of t^m on both sides, we obtain the following addition theorem for the ω_{-m-1}:

$$(5.11)\qquad (-1)^{m+1}\,\omega_{-m-1}(\xi^*,\eta^*) = \sqrt{1+A^2}\ \sum_{s=0}^{\infty}(-1)^{s+1}\,\omega_{-s-1}(\xi,\eta)\,C_s(A),$$

where

$$(5.12)\qquad C_s(A) = (-A)^{m+s}\sum_{n=0}^{m}\frac{(s+n-1)!}{(s-1)!\,n!}\ {}_2F_1(-n,-s;1-n-s;-A^{-2}).$$

In (5.12), the term

$$\frac{(s+n-1)!}{(s-1)n!}$$

has to be replaced by unity if $s = n = 0$.

It can be shown that the series on the right-hand side of (5.11) converges for $|A| < 1$. A similar addition theorem can be derived for the ω_m, $m = 0, 1, 2, \ldots$.

6. Expansion problem for ω_ν

We now wish to investigate the possibility of obtaining an expansion for the one-valued regular solutions ω_ν for an arbitrary parameter ν in terms of the solutions ω_n, $n = 0, \pm 1, \pm 2, \ldots$.

We begin with the formula

$$\sqrt{\pi/2} \cdot \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \omega_\nu(\xi,\eta) \frac{t^{-\nu-1}}{\sin(-\nu\pi)} d\nu = G(\xi,\eta,t)$$

$$(6.1) \qquad = \frac{1}{\sqrt{1+t^2}} \exp\left\{ ik \left[\frac{1-t^2}{1+t^2} \cdot \frac{\xi^2-\eta^2}{2} + \frac{2t}{1+t^2} \cdot \xi\eta\right]\right\} .$$

By inversion we obtain from the above Mellin integral

$$\omega_\nu(\xi,\eta) = \sqrt{2/\pi} \int_0^\infty G(\xi,\eta,t)\, t^\nu\, dt$$

$$(6.2) \qquad = \sqrt{2/\pi} \left(\int_0^1 G(\xi,\eta\, t)t^\nu dt + \int_1^\infty G(\xi,\eta,t)t^\nu dt \right)$$

$$= \sqrt{2/\pi} \left(\int_0^1 \sum_0^\infty (-1)^n \omega_{-n-1}(\xi,\eta) t^{n+\nu} dt \right.$$

$$\left. + \int_1^\infty \sum_0^\infty (-1)^n \omega_n(\xi,\eta) t^{-n-1+\nu}\, dt \right),$$

where $-1 < \nu < 0$. Now, if an expansion of the kind contemplated were possible, then the expression in parenthesis in the last member of equation (6.2) would be equal to

$$(6.3) \qquad \sum_0^\infty (-1)^n \frac{\omega_{-n-1}(\xi,\eta)}{n+\nu+1} + \sum_0^\infty (-1)^n \frac{\omega_n(\xi,\eta)}{-n+\nu} .$$

But the asymptotic expansion of the generic term of each series, and the application of the Cauchy ratio test, shows that the expression (6.3) diverges.

Part II. A Diffraction Problem

1. Formulation of the problem

We consider the two-dimensional wave equation

$$\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} + k^2(\xi^2 + \eta^2)u = 0 \tag{1.1}$$

in the coordinates of the parabolic cylinder. We are interested in the diffracted wave u due to a source at a point on the axis of a perfectly conducting parabolic cylinder. Actually, we regard the source as a line source generated by a current in a wire parallel to the generators of the cylinder.

The conditions imposed upon u are the following:

(a) On the parabolic cylinder the values of u are the negatives of the values of the field due to the source; i.e.,

$$u(\xi, \eta_o) = -H_o^{(1)}(kr), \tag{1.2}$$

where $r^2 = (\xi\eta - \xi_o\eta_o)^2 + \frac{(\xi^2 - \eta^2)^2}{4}$, and $H_o^{(1)}$ is the Hankel function of the first kind.

(b) In the region $|\eta| < |\eta_a|$, the function u should satisfy the wave equation and should be regular. $\eta = \pm \eta_a$ represents the boundary of the parabolic cylinder.

(c) The function u should satisfy the Sommerfeld radiation condition. The correct formulation of this condition for our problem is due to Magnus [4]. It states, in the case k = 1, that for $\xi \to \infty$ and for a sufficiently small $\varepsilon > 0$,

$$\left| \int_{-\eta_a}^{\eta_a} \bar{u} \left\{ \xi^{-1} \frac{\partial u}{\partial \xi} - iu \right\} d\eta \right| \leq \left| \xi^{1-\varepsilon} \int_{-\eta_a}^{\eta_a} |u|^2 \, d\eta \right| , \tag{1.3}$$

where $\bar{u}$ is the complex conjugate of u.

A function u satisfying these conditions can be derived from a formula due to Buchholz (see [2], p. 177):

$$H_0^{(1)}(kr) = \frac{i\sqrt{2}}{\pi}\,\frac{1}{2\pi i}\int_{-\sigma-i\infty}^{-\sigma+i\infty}\Gamma\left(-s+\frac{1}{4}\right)\Gamma\left(s+\frac{1}{4}\right)\sqrt{2}\cdot D_{-2s-1/2}(\sqrt{-2ik}\;\eta)\left\{\begin{array}{ll} E^{(0)}_{2s-1/2}(\sqrt{-2ik}\;\xi)\cdot D_{2s-1/2}(\sqrt{-2ik}\;\xi_0) & \xi_0<\xi \\ E^{(0)}_{2s-1/2}(\sqrt{-2ik}\;\xi_0)\cdot D_{2s-1/2}(\sqrt{-2ik}\;\xi) & \xi_0>\xi \end{array}\right\}ds \tag{1.4}$$

where D_ν are the parabolic cylinder functions which we have discussed in the first part of the paper, and the $E_\nu^{(0)}$ are defined by

$$E_\nu^{(0)}(Z) = 2^{1/2}\,e^{-z^2/4}\;{}_1F_1\left(-\frac{\nu}{2};\frac{1}{2};\frac{z^2}{2}\right).$$

From (1.4) we can write for the function u, if $\xi_0 > \xi$,

$$u(\xi,\eta;\xi_0,0;\eta_a) = -\frac{1}{\pi^2}\int_{-\sigma-i\infty}^{-\sigma+i\infty}\Gamma\left(-s+\frac{1}{4}\right)\Gamma\left(s+\frac{1}{4}\right)\cdot E^{(0)}_{2s-1/2}(\sqrt{-2ik}\;\xi)\cdot D_{2s-1/2}(\sqrt{-2ik}\;\xi_0)\cdot D_{-2s-1/2}(\sqrt{-2ik}\;\eta_a)\cdot\frac{E^{(0)}_{-2s-1/2}(\sqrt{-2ik}\;\eta)}{E^{(0)}_{-2s-1/2}(\sqrt{-2ik}\;\eta_a)}\,ds\,. \tag{1.5}$$

In formulas (1.4) and (1.5), $s = \sigma + i\tau$. In (1.4), $|\sigma| < \frac{1}{4}$; in (1.5), $0 < \sigma < \frac{1}{4}$.

It remains to show that the function u in (1.5) satisfies the conditions (a), (b), and (c).

For the boundary condition (a) this is easily proved by putting $\eta = \eta_a$ in (1.5). For the investigation of the other conditions it will be more convenient to confine ourselves to the case in which $\xi_o = \eta_o = 0$, thereby avoiding unnecessary computational difficulties. The expression (1.5) then becomes

$$u = \frac{-\sqrt{2}}{\pi^2} \int_{\sigma - i\infty}^{\sigma + i\infty} \Gamma(-s+\tfrac{1}{4})\Gamma(s+\tfrac{1}{4}) D_{2s-1/2}(\xi\sqrt{-2ik}\,)$$

(1.6)

$$\cdot \frac{E^{(0)}_{-2s-1/2}(\eta\sqrt{-2ik}\,)}{E^{(0)}_{-2s-1/2}(\eta_a\sqrt{-2ik})}\, ds.$$

The region which we consider is the interior of the parabola, and this is described by the inequalities

$$-\eta_a < \eta < \eta_a, \quad \xi \geq 0.$$

First we study the question of regularity for this region. This involves the properties of finiteness and differentiability of the function u. The finiteness is a consequence of the behavior of the expression

$$(1.7) \qquad D_{2s-1/2}(\xi\sqrt{-2ik}\,) \cdot \frac{E^{(0)}_{-2s-1/2}(\eta\sqrt{-2ik}\,)}{E^{(0)}_{-2s-1/2}(\eta_a\sqrt{-2ik}\,)} \cdot \Gamma(-s+\tfrac{1}{4})\Gamma(s+\tfrac{1}{4})$$

for large s. We now require asymptotic expansions of the functions $D_\nu(z)$ and $E^{(0)}_\nu(z)$ for large ν. From [2], page 40, formula (27,a) we have

$$(1.8) \qquad E^{(0)}_{-2s-1/2}(z) = \sqrt{2}\, e^{-z^2/4} \cdot {}_1F_1\left(s+\frac{1}{4}\,;\,\frac{1}{2}\,;\,\frac{z^2}{2}\right).$$

Following the discussion in [1], vol. I, page 279, especially formula (12), we have for $|s| \to \infty$:

$$E^{(0)}_{-2s-1/2}(\eta\sqrt{-2ik}\,) \sim \sqrt{2}\, e^{ik\eta^2/2} \sqrt{2}\, \Gamma(\tfrac{1}{2}) e^{-ik\eta^2/2}(-s)^{1/2}(-ik\eta^2)^{1/2}$$

(1.8)'

$$\cdot \sqrt{2/\pi}\, \cos\left[\eta\sqrt{iks}\right].$$

We choose our path C of integration in the right half-plane as the union of the two paths:

$$s = 2\tau + i\,\tau^2 \qquad 0 \leqq \tau < \infty,$$

$$s = -2\tau - i\,\tau^2 \qquad -\infty < \tau \leqq 0 \;.$$

For $\tau < 0$, $\cos\left[\eta_a \sqrt{iks}\right] \sim \cos\left[(\tau + i)\,\eta_a \sqrt{k}\right]$. The absolute value of this expression, and therefore of the denominator, is bounded away from zero. Thus, for $\tau < 0$, the quotient of the two $E_\nu^{(0)}$-functions is bounded, while for $\tau > 0$, it tends to zero for $|\eta| < \eta_a$.

As for $D_\nu(z)$, we have

$$(1.9) \qquad D_\nu(z) = 2^{1/2(\nu-1)}\, e^{-z^2/2}\, \Psi\left(\frac{1}{2} - \frac{\nu}{2}\,;\, \frac{3}{2}\,;\, \frac{z^2}{2}\right).$$

From [1], vol. I, page 279, (8), with $\nu = 2s - \frac{1}{2}$, $a = \frac{1}{2} - \frac{\nu}{2}$, $\kappa = s$, and $x = -ik\xi^2$, we obtain, since $0 < \arg \kappa < \frac{\pi}{2}$,

$$D_{2s-1/2}(\xi\sqrt{-2ik}\,) \sim 2^{s-3/4} \cdot e^{ik\xi^2} \cdot \xi\sqrt{-2ik} \cdot s^{-1/4+s} \cdot (-ik\xi^2)$$

$$\cdot\, e^{-s-(1/2)ik\xi^2} \cdot \sqrt{2}\, \cos\left[\pi s - 2\sqrt{-sik\xi^2} - \frac{\pi}{4}\right].$$

Now $s^{-1/4+s} = s^{-1/4} \cdot e^{s \log s}$. For $\tau > 0$

$$\operatorname{Re}(s \log s) \sim -\frac{\pi}{2}\,\tau^2 + 4\,\tau \log \tau \;.$$

For $\tau < 0$,

$$\operatorname{Re}(s \log s) \sim -\frac{\pi}{2}\,\tau^2 + \text{terms of order } \tau \text{ or } \tau \log|\tau|.$$

Next we must consider the factor

$$\cos\left[\pi s - 2\sqrt{-sik} \cdot \xi - \frac{\pi}{4}\right],$$

$$\xi\sqrt{k} \cdot \sqrt{-is} \sim i\tau\,\xi\sqrt{k} \quad \text{for} \quad \tau < 0.$$

Hence we have, apart from factors whose absolute value is bounded:

$$\cos\left[\pi s - 2\sqrt{-sik\xi^2} - \frac{\pi}{4}\right] \sim \exp\left\{\pi\tau^2 + 2|\tau|\,\xi\sqrt{k}\right\}, \quad \text{for } \tau < 0;$$

and

$$\cos\left[\pi s - 2\sqrt{-sik\xi^2} - \frac{\pi}{4}\right] \sim \exp\left\{\pi\,\tau^2\right\} \quad \text{for} \quad \tau > 0.$$

Now consider $\Gamma(-s + \frac{1}{4}) \cdot \Gamma(s + \frac{1}{4})$; we have

$$\Gamma(-s + \tfrac{1}{4})\,\Gamma(s + \tfrac{1}{4}) \sim 2\pi \exp\left\{ s \log\left(\frac{s}{-s}\right) - \frac{1}{4}\log(-s^2)\right\} .$$

If Im s > 0, then $\log(\frac{s}{-s}) = i\pi$; if Im s < 0, then $\log(\frac{s}{-s}) = -i\pi$. Therefore,

$$\Gamma(-s + \tfrac{1}{4})\,\Gamma(s + \tfrac{1}{4}) \sim 2\pi \exp\left\{is(\operatorname{sgn} \operatorname{Im} s)\pi\right\} \cdot (-s^2)^{-1/4}$$

$$(1.10) \qquad 2\pi s^{-1/2}\, e^{-i\pi/4} \circ \exp \begin{cases} (-\tau^2 + 2\tau i)\pi & \text{if } \tau > 0 \\ (-\tau^2 - 2i\tau)\pi & \text{if } \tau < 0 \end{cases}$$

In absolute value, therefore,

$$(1.11) \qquad \left|\Gamma(-s + \tfrac{1}{4})\,\Gamma(s + \tfrac{1}{4})\right| \sim \frac{2\pi}{\sqrt{|s|}} \cdot e^{-\pi\tau^2} .$$

Collecting all expressions we obtain for (1.7), apart from factors whose absolute value is 1:

$$(1.12) \quad 2\pi s^{-\frac{1}{2}} \cdot e^{-\pi\tau^2} \cdot s^{-\frac{1}{4}} \cdot e^{4\tau\log\tau} \cdot e^{\pi\tau^2} \cdot 2^{2\tau} \cdot e^{-2\tau} \cdot e^{-\pi/2\,\tau^2}\, \frac{E^{(0)}_{-2s-1/2}(\eta\sqrt{-2ik})}{E^{(0)}_{-2s-1/2}(\eta_a\sqrt{-2ik})}$$

which tends strongly to zero as $\tau \to \infty$.

Next, we study the property of differentiability of u as given by (1.6). From $x = (\xi^2 - \eta^2)/2$ and $y = \xi\eta$, it follows that

$$(1.13) \qquad \begin{aligned} \frac{\partial u}{\partial x} &= \left(\xi \frac{\partial u}{\partial \xi} - \eta \frac{\partial u}{\partial \eta}\right)\frac{1}{\xi^2 + \eta^2} , \\ \frac{\partial u}{\partial y} &= \left(\eta \frac{\partial u}{\partial \xi} + \xi \frac{\partial u}{\partial \eta}\right)\frac{1}{\xi^2 + \eta^2} . \end{aligned}$$

There is no problem for $\xi > 0$. But for $\xi = 0$, $\frac{\partial u}{\partial \xi}$ really means $\frac{\partial u}{\partial |\xi|}$, because, by definition, $\xi \geq 0$. We assume $\eta \neq 0$. For the existence and continuity of the derivatives $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ on $\xi = 0$ it is necessary that

$$(1.14) \qquad \frac{\partial u}{\partial \xi} = \frac{\partial u}{\partial |\xi|} = 0.$$

To prove this we observe that

$$\left.\frac{\partial u}{\partial \xi}\right|_{\xi=0} = -\frac{1}{\pi^2}\int_C \Gamma(s+\tfrac{1}{4})\Gamma(-s+\tfrac{1}{4}) \cdot \frac{2^{s-1} \cdot \Gamma(-\tfrac{1}{2})}{\Gamma(-s+\tfrac{1}{4})}$$

$$(1.15) \qquad \cdot \frac{E^{(0)}_{-2s-1/2}(\eta\sqrt{-2ik})}{E^{(0)}_{-2s-1/2}(\eta_a\sqrt{-2ik})}\, ds.$$

But since the integrand in (1.15) has no poles to the right of the path of integration, condition (1.14) is proved.

Lastly, we must consider the radiation condition. We shall not go into detail. We merely remark that we can base our work on a formula of Magnus, (see [5], page 349, formula (19)). This formula can be obtained by closing the path of integration with respect to s to the left. We then obtain an expansion of

$$H_0^{(1)}(kr) + u$$

in terms of a sum of products of the type

$$D_{2s_\ell - 1/2}(\xi\sqrt{-2ik}) \cdot E^{(0)}_{-2s_\ell - 1/2}(\eta\sqrt{-2ik}),$$

where the s_ℓ are the zeros of $E^{(0)}_{-2s-1/2}(\eta_a\sqrt{-2ik}) = 0$ for a fixed η_a.

2. Remarks on the case of short wavelength

In this section we are interested in studying the behavior of

$$u = -\frac{1}{\pi^2}\int_C \Gamma(\tfrac{1}{4}+s)\Gamma(\tfrac{1}{4}-s)\cdot E^{(0)}_{2s-1/2}(\xi\sqrt{-2ik})\cdot D_{2s-1/2}(\xi_o\sqrt{-2ik})$$

$$(2.1)\qquad \cdot D_{-2s-1/2}(\eta_a\sqrt{-2ik})\cdot\frac{E^{(0)}_{-2s-1/2}(\eta\sqrt{-2ik})}{E^{(0)}_{-2s-1/2}(\eta_a\sqrt{-2ik})}\,ds\,,$$

for large $k = 2\pi/\lambda$. This requires the asymptotic expansion of $D_\nu(z)$ and $F_\nu^{(0)}(z)$ for large z. We assume that the asymptotic expansion of these functions for $|z| \to \infty$, if inserted into the integral, will give a reasonable approximation to the value of u for $k \to \infty$. We have, for $|z| \to \infty$:

$$(2.2)\qquad D_\nu(z) \sim e^{-z^2/4}\cdot z^\nu\,,$$

$$E_\nu^{(0)}(z) = \sqrt{2}\cdot e^{-z^2/4}\cdot {}_1F_1\left(-\frac{\nu}{2};\frac{1}{2};\frac{z^2}{2}\right)$$

$$(2.3)\qquad \sim\sqrt{2}\cdot e^{-z^2/4}\cdot e^{i\pi\nu/2}\,\frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1+\nu}{2})}\cdot\left(\frac{z^2}{2}\right)^{\nu/2}$$

$$+\sqrt{2}\cdot e^{z^2/4}\cdot\left(\frac{z^2}{2}\right)^{-\nu+1/2}\,\frac{\Gamma(\frac{1}{2})}{\Gamma(-\frac{\nu}{2})}\,.$$

Since we are interested in the case in which the line source lies on the axis of the parabola, we take $\eta = 0$. Then we have

$$u \sim -\frac{\sqrt{2}}{\pi^2}\int_C \Gamma(\tfrac{1}{4}+s)\Gamma(\tfrac{1}{4}-s)\exp\left[\frac{ik}{2}(\xi_o^2+\eta_a^2)\right]\cdot(\xi_o\sqrt{-2ik})^{2s-1/2}(\eta_a\sqrt{-2ik})^{-2s-1/2}$$

$$(2.4)\qquad \cdot\frac{\exp\left[-\frac{ik\xi^2}{2}\right]\exp\left[i\pi(s-\frac{1}{4})\right]\frac{\sqrt{\pi}}{\Gamma(s+\frac{1}{4})}(-ik\xi^2)^{s-1/4}+\exp\left[\frac{ik\xi^2}{2}\right]\frac{\sqrt{\pi}}{\Gamma(\frac{1}{4}-s)}(-ik\xi^2)^{-s-1/4}}{\exp\left[-\frac{ik\eta_a^2}{2}\right]\exp\left[-i\pi(s+\frac{1}{4})\right]\frac{\sqrt{\pi}}{\Gamma(\frac{1}{4}-s)}(-ik\eta_a^2)^{-s-1/4}+\exp\left[\frac{ik\eta_a^2}{2}\right]\frac{\sqrt{\pi}}{\Gamma(\frac{1}{4}+s)}(-ik\eta_a^2)^{s-1/4}}\,ds.$$

We close the path of integration to the right and apply the residue theorem (simple poles at $s = \frac{1}{4} + n$, $n = 0, 1, 2, \ldots$), obtaining

$$(2.5) \qquad u \sim \frac{2\sqrt{2}\, i}{\pi} \exp\left[\frac{ik}{2}(\xi_o^2 - \xi^2)\right] \frac{1}{\eta_a\sqrt{-2ik}} \sum_{n=0}^{\infty} \frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)} w^n ,$$

where $w = \left(\frac{\xi_o \xi}{\eta_a^2}\right)^2$; $\frac{\xi_o \xi}{\eta_a^2}$ = coordinate of the cusp of the caustic of reflected rays. Since

$$\sum_{n=0}^{\infty} \frac{(1/2)_n}{n!} w^n = (1 - w)^{1/2},$$

the expression (2.5) becomes

$$(2.6) \qquad u \sim \frac{2\sqrt{2}\, i}{\pi} \frac{\exp\left[\frac{ik}{2}(\xi_o^2 - \xi^2)\right]}{\eta_a \sqrt{-2ik}} (1 - w)^{-1/2} .$$

On the axis of the parabola, between the cusp of the caustic (in the limiting case of geometrical optics) and the vertex of the parabola, the field is everywhere of the same order of magnitude with respect to k. The fact that our formula breaks down for $w = 1$ indicates that this order of magnitude changes at the cusp.

In the integral (2.1), the behavior of $D_\nu(z)$ and $E_\nu^{(0)}(z)$ should be studied for large parameter ν as well as for large $|z|$. However, the results of the preceding section suggest that the contribution for large ν may be negligible.

Acknowledgement

The author wishes to express his gratitude to Professor Wilhelm Magnus for his advice and encouragement in the preparation of this report. We are indebted also to the Evans Signal Laboratory for making this work possible.

Bibliography

[1] Erdelyi, A., et.al. - Higher transcendental functions; McGraw-Hill, 1953.

[2] Buchholz, H. - Die Konfluente Hypergeometrische Funktion; Springer, 1953.

[3] Grinberg, G., et.al - Wave problem for the parabolic mirror; Dok. Akad. Nauk SSSR, 95, 961-63 (1954).

[4] Magnus, W. - Über eine Randwertaufgabe der Wellengleichung für den parabolischen Zylinder; Jahresb. Deutsch. Math.-Verein., 50, 140-161 (1940).

[5] Magnus, W. - Zur Theorie des Zylindrisch-parabolischen Spiegels; Z. Phys., 118, 343-356 (1941).

[6] Meixner, J., and Schäfke, F.W. - Mathieusche Funktionen und Spheroidfunktionen; Springer, 1954.

DISTRIBUTION LIST FOR RESEARCH REPORTS

Contract No. AF 18(600)-367

(ONE copy unless otherwise noted)

The Air University Libraries
Maxwell Air Force Base, Alabama

Applied Mathematics and Statistics Lab.
Stanford University
Stanford, California

Department of Mathematics
University of California
Berkeley, California

Commander
Air Force Flight Test Center
ATTN: Technical Library
Edwards Air Force Base, California

The Rand Corporation
Technical Library
1700 Main Street
Santa Monica, California

Director, Office of Advanced Studies
Air Force Office of Scientific Research
Post Office Box 2035
Pasadena 2, California

Commander
Western Development Division
ATTN: WDSIT
P.O. Box 262
Inglewood, California

Department of Mathematics
Yale University
New Haven, Connecticut

Commander
Air Force Armament Center
ATTN: Technical Library
Eglin Air Force Base, Florida

Commander
Air Force Missile Test Center
ATTN: Technical Library
Patrick Air Force Base, Florida

Department of Mathematics
Northwestern University
Evanston, Illinois

Institute for Air Weapons Research
Museum of Science and Industry
University of Chicago
Chicago 37, Illinois

Department of Mathematics
University of Chicago
Chicago 37, Illinois

Department of Mathematics
University of Illinois
Urbana, Illinois

Institute for Fluid Dynamics & Applied Math.
University of Maryland
College Park, Maryland

Mathematics and Physics Library
The Johns Hopkins University
Baltimore, Maryland

Department of Mathematics
Harvard University
Cambridge, Massachusetts

Commander
Air Force Cambridge Research Center
ATTN: Geophysics Research Library
Cambridge 39, Massachusetts

Commander
Air Force Cambridge Research Center
ATTN: CRRBA
Cambridge 39, Massachusetts

Commander
Air Force Cambridge Research Center
ATTN: Electronic Research Library
Cambridge 39, Massachusetts

Institute of the Aeronautical Sciences
ATTN: Library Acquisitions
2 East 64th Street
New York 21, N.Y.

Department of Mathematics
Columbia University
New York 27, New York
ATTN: Professor B.O. Koopman

Commander
Rome Air Development Center
ATTN: Technical Library
Griffiss Air Force Base
Rome, New York

Institute of Statistics
North Carolina State College of A & E
Raleigh, North Carolina

Department of Mathematics
University of North Carolina
Chapel Hill, North Carolina

Office of Ordnance Research
Box CM
Duke Station
Durham, North Carolina

Department of Mathematics
Duke University
Duke Station
Durham, North Carolina

Commander
Air Technical Intelligence Center
ATTN: ATIAE-4
Wright-Patterson Air Force Base, Ohio

Commander
Wright Air Development Center
ATTN: Technical Library
Wright-Patterson Air Force Base, Ohio

(2)Commander
Wright Air Development Center
ATTN: ARL Technical Library, WCRR
Wright-Patterson Air Force Base, Ohio

(2)Commandant
USAF Institute of Technology
ATTN: Technical Library, MCLI
Wright-Patterson Air Force Base, Ohio

(5)Chief, Document Service Center
Armed Services
Technical Information Agency
Knott Building
Dayton 2, Ohio

Department of Mathematics
Carnegie Institute of Technology
Pittsburgh, Pennsylvania

Department of Mathematics
University of Pennsylvania
Philadelphia, Pennsylvania

Commander
Arnold Engineering Development Center
ATTN: Technical Library
Tullahoma, Tennessee

Defense Research Laboratory
University of Texas
Austin, Texas

Department of Mathematics
Rice Institute
Houston, Texas

Commander
Air Force Personnel & Training Research
Center
ATTN: Technical Library
Lackland Air Force Base
San Antonio, Texas

Department of Mathematics
University of Wisconsin
Madison, Wisconsin

(2)Commander
Air Force Office of Scientific Research
ATTN: SROAM
Washington 25, D.C.

Commander
Air Force Office of Scientific Research
ATTN: SRRI
Washington 25, D.C.

Dr. Y.W. Chen
2645 Stuart Street
Berkeley, California

Willow Run Research Center
University of Michigan
Ypsilanti, Michigan

Department of Mathematics
Folwell Hall
University of Minnesota
Minneapolis, Minnesota

Department of Mathematics
Institute of Technology
Engineering Building
University of Minnesota
Minneapolis, Minnesota

Department of Mathematics
Washington University
Saint Louis 5, Missouri

Department of Mathematics
University of Missouri
Columbia, Missouri

Commander
Strategic Air Command
ATTN: Operations and Analysis
Offutt Air Force Base
Omaha, Nebraska

The James Forrestal Research Center Library
Princeton University
Princeton, New Jersey

Library
Institute for Advanced Study
Princeton, New Jersey

Department of Mathematics
Fine Hall
Princeton University
Princeton, New Jersey

Commander
Holloman Air Development Center
ATTN: Technical Library
Holloman Air Force Base, New Mexico

Commander
Air Force Special Weapons Center
ATTN: Technical Library
Kirtland Air Force Base
Albuquerque, New Mexico

Prof. J. Wolfowitz
Mathematics Department
White Hall
Cornell University
Ithaca, New York

Department of Mathematics
Syracuse University
Syracuse, New York

Commander
Rome Air Development Center
ATTN: Intelligence and Analysis Section of
Electronic Warfare
Rome, New York

Department of Mathematical Statistics
Fayerweather Hall
Columbia University
New York 27, New York
ATTN: Dr. Herbert Robbins

Human Factors Operations Research Labs.
Air Research and Development Command
Bolling Air Force Base
Washington 25, D.C.

(2)Chief of Naval Research
Department of the Navy
ATTN: Code 432
Washington 25, D.C.

Department of Commerce
Office of Technical Services
Washington 25, D.C.

Director National Security Agency
ATTN: Dr. H.H. Campaigne
Washington 25, D.C.

Library
National Bureau of Standards
Washington 25, D.C.

National Applied Mathematics Labs.
National Bureau of Standards
Washington 25, D.C.

Headquarters, USAF
Director of Operations
Washington 25, D.C.
ATTN: Operations Analysis Division, AFOOP

Commander
European Office ARDC
60 Rue Ravenstein
Brussels, Belgium

Naval Electronics Laboratory
San Diego 52, California

Research Laboratory of Electronics
Massachusetts Institute of Technology
Rm. 20B-221
Cambridge 39, Massachusetts

Signal Corps Engineering Laboratories
Technical Documents Center
Evans Signal Laboratory
Belmar, New Jersey

Commanding General
Signal Corps Engineering Laboratories
ATTN: Technical Reports Library
Fort Monmouth, New Jersey

Chief of Staff
HQ United States Air Force
Pentagon, Washington 25, D.C.
ATTN: AFMRD - 5

Office of Chief Signal Officer
Signal Plans and Operations Division
Com. Liaison Br., Radio Prop. Section
The Pentagon, Washington 25, D.C.
ATTN: SIGOL - 2, Rm. 20

Naval Research Laboratory
Washington 25, D.C.
ATTN: Technical Data Section

Central Radio Propagation Laboratory
National Bureau of Standards
Washington 25, D.C.
ATTN: Technical Reports Library

Air Force Development Field Representative
Naval Research Laboratory
Code 1110
Washington 25, D.C.

Ballistics Research Laboratory
Aberdeen Proving Ground
Aberdeen, Maryland
ATTN: Dr. Keats Pullen

Massachusetts Institute of Technology
Lincoln Laboratory
P.O. Box 390
Cambridge 39, Massachusetts
ATTN: Dr. T.J. Carroll

Mr. Keeve Siegel
Willow Run Research Center
Willow Run Airport
Ypsilanti, Michigan

Willow Run Research Center
Willow Run Airport
Ypsilanti, Michigan
ATTN: Dr. C.L. Dolph

Director, Naval Research Lab.
Code 3480
Washington 25, D.C.
ATTN: Dr. L.C. van Atta

Director, Naval Research Lab.
Washington 25, D.C.
ATTN: Mr. Robert E. Roberson

Director Naval Research Lab.
Washington 25, D.C.

National Bureau of Standards
Computation Laboratories
Washington 25, D.C.
ATTN: Dr. John Todd

Dr. A.G. McNish
National Bureau of Standards
Washington 25, D.C.

Dr. Derrick H. Lehmer
Department of Mathematics
University of California
Berkeley, California

Dr. Joseph Kaplan
Department of Physics
University of California
Los Angeles, California

Division of Electrical Engineering
Electronics Research Laboratory
University of California
Berkeley 4, California
ATTN: Dr. Samuel Silver

Dr. A. Erdelyi
California Institute of Technology
1201 E. California Street
Pasadena, California

Dr. Robert Kalaba
Electronics Division
Rand Corporation
Santa Monica, California

Stanford Research Institute
Stanford, California
ATTN: Dr. J.V.N. Granger
head, Radio Systems Laboratory

Dr. Vic Twersky
Electronics Defense Laboratory
Box 205
Mountain View, California

Hughes Aircraft Company
Research and Development Library
Culver City, California
ATTN: M. Bodner

National Bureau of Standards
Boulder Laboratories
Boulder, Colorado
ATTN: Library

Georgia Institute of Technology
Engineering Experimental Station
Atlanta, Georgia
ATTN: Dr. James E. Boyd

Applied Physics Laboratory
The Johns Hopkins University
8621 Georgia Avenue
Silver Spring, Maryland
ATTN: Mr. F.T. McClure

Dr. Donald E. Kerr
Department of Physics
The Johns Hopkins University
Baltimore 18, Maryland

Dr. A. Weinstein
Institute of Fluid Dynamics and Applied Math.
University of Maryland
College Park, Maryland

Professor J.A. Pierce
Harvard University
Cambridge 38, Massachusetts

Mrs. Marjorie L. Cox, Librarian
Technical Reports Collection
Room 303 A, Pierce Hall
Harvard University
Cambridge 38, Massachusetts

Dr. F.M. Wiener
Bolt Beranek and Newman Inc.
16 Eliot Street
Cambridge 38, Massachusetts

Dr. Arthur A. Oliner
Microwave Research Institute
Polytechnic Institute of Brooklyn
55 Johnson Street
Brooklyn, New York

Professor J.H. Mulligan
School of Engineering
New York University
New York, N.Y.

Dr. H.G. Booker
School of Electrical Engineering
Cornell University
Ithaca, New York

Professor Bernard Epstein
Department of Mathematics
University of Pennsylvania
Philadelphia 4, Pennsylvania

Professor Albert Heins
Carnegie Institute of Technology
Pittsburgh, Pennsylvania

Professor Fred A. Ficken
University of Tennessee
Knoxville, Tennessee

Miss Barbara C. Grimes, Librarian
Federal Communications Commission
Washington 25, D.C.

Carnegie Institute of Washington
Department of Terrestrial Magnetism
5241 Broad Branch Road, N.W.
Washington 15, D.C.
ATTN: Library

Mathematical Reviews
80 Waterman Street
Brown University
Providence, Rhode Island

Mr. Martin Katzin
154 Fleetwood Terrace
Silver Spring, Maryland

Prof. B.H. Bissinger
Lebanon Valley College
Annville, Pennsylvania

Applied Physics Laboratory
The Johns Hopkins University
8621 Georgia Avenue
Silver Spring, Maryland
ATTN: Dr. B.S. Gourary

Dr. Charles H. Papas
Department of Electrical Engineering
California Institute of Technology
Pasadena, California

Dr. Rodman Doll
615 Lawrence Street
Ann Arbor, Michigan

Mr. Calvin H. Wilcox
California Institute of Technology
Pasadena 4, California

Technical Director
Combat Development Dept.
Army Electronic Proving Ground
Fort Huachuca, Arizona

National Bureau of Standards
Boulder, Colorado
ATTN: Dr. W.R. Gallet

Dr. William Bauer
Antenna Research Laboratory
Republic Aviation Corp.
Farmingdale, L.I., N.Y.

Mrs. A.M. Gray
Engineering Library
Plant 5
Grumman Aircraft Corp.
Bethpage, L.I., N.Y.

Professor Bernard Friedman
Department of Mathematics
University of California
Berkeley, California

Dr. Jane Scanlon
284 South Street
Southbridge, Massachusetts

Dr. Solomon L. Schwebel
8957 Eames Avenue
Northridge, California

Dr. Bernard Lippmann
Microwave Laboratory
55 Johnson Street
Brooklyn, N.Y.

University of Minnesota
The University Library
Minneapolis 14, Minnesota
ATTN: Exchange Division

Lincoln Laboratory
Massachusetts Institute of Technology
P.O. Box 73
Lexington 73, Massachusetts
ATTN: Dr. Shou Chin Wang, Rm. C-351

Mr. K.S. Kelleher
Section Head
Melpar, Inc.
3000 Arlington Boulevard
Falls Church, Virginia

Technical Research Group
17 Union Square West
New York 3, N.Y.
ATTN: Dr. L. Goldmuntz

Antenna Research Section
Microwave Laboratory
Hughes Aircraft Company
Culver City, California
ATTN: Dr. Richard B. Barrar

Institute of Fluid Dynamics & Applied Math.
University of Maryland
College Park, Maryland
ATTN: Dr. Elliot Montroll

Brandeis University
Waltham, Massachusetts
ATTN: Library

General Electric Company
Microwave Laboratory
Electronics Division
Stanford Industrial Park
Palo Alto, California
ATTN: Library

Dr. W.A. Dolid
Hughes Research Laboratories
Hughes Aircraft Company
Culver City, California
Bldg. 12, Rm. 2529

Dr. Jerry Shmoys
Dept. of Electrical Engineering
Brooklyn Polytechnic
85 Livingston Street
Brooklyn, N.Y.

Dr. I. Kolodner
Dept. of Mathematics
University of New Mexico
Albuquerque, New Mexico

Dr. John B. Smyth
Smyth Research Associates
3930 4th Avenue
San Diego 3, California

Dr. George G. Weill
Electrical Engineering
California Institute of Technology
Pasadena, California

Naval Research Laboratory
Washington 25, D.C.
ATTN: Dr. Henry J. Passerini
Code 5278 A

Dr. George Kear
P.O. Box 64
Burnt Hills, N.Y.

California Institute of Technology
Electrical Engineering
Pasadena, California
ATTN: Dr. Zohrab A. Kaprielian

Dr. Nathan Marcuvitz
Brooklyn Polytechnic
85 Livingston Street
Brooklyn, N.Y.

[illegible] 8 [illegible]

DATE DUE

GAYLORD			PRINTED IN U.S.A.

NYU c.2
BR-
19 Epstein

On the functions of the
parabolic cylinder.

NYU c.2
BR-
19 Epstein

AUTHOR
On the functions of the

TITLE
parabolic cylinder.

DATE DUE	BORROWER'S NAME	ROOM NUMBER

N.Y.U. Courant Institute of
Mathematical Sciences
251 Mercer St.
New York 12, N. Y.